# TUT 3

Uniqueness thm:

$$
① \begin{cases} y'' + p(t)y' + q(t)y = g(t) \\ y(t_0) = y_0 \\ y'(t_0) = y_0' \end{cases}
$$

where $p, q, g$ are cts on an open interval I that contains the point $t_0$,

Then $\exists!$ $y = \phi(t)$ satisfies ① throughout I.

Consider $L(y) = y'' + py' + qy = 0$

Def: Suppose $y_1, y_2$ are the solution to $L(y) = 0$, then $W(y_1, y_2)(t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

is called the Wronskian of $y_1$ and $y_2$

Actually, we want to make use of Wronskian to see if our solutions $y_1, y_2$ to $L(y) = 0$ are "different".

Thm: If $y_1, y_2$ are 2 solutions of $L(y)=0$, and $\exists t_0$ s.t. $W(y_1,y_2)(t_0) \neq 0$, then

$y = c_1 y_1 + c_2 y_2$ includes all the solutions of $L(y)=0$. for arbitrary $c_1, c_2$.

Abel thm: If $y_1, y_2$ are the solution to $L(y)=0$, where, $p, q$ are cts, then Wronkian,

$W(y_1,y_2)(t) = Ce^{-\int p}$.

Remark: Abel thm implies the wronkian of the solution of an ODE must be alway zero or never zero. ($C=0$ or $C \neq 0$).

Thm: Let $y_1, y_2$ be the solutions of $L(y)=0$, where $p, q$ are cts,

$y_1, y_2$ are linearly independent iff $W \neq 0$.

Remark: It is not true if $y_1, y_2$ are not the solution of a ODE.

Nonhomogeneous equation:

$$L(y) = y'' + p(t)y' + q(t)y = g(t). \quad -②$$

there is no a formula to solve this general nonhomogeneous equation.

We deal with the case of constant p, q.

To solve $y'' + ay' + by = g(t)$

Step ①: find the general solution to the homogeneous equation $y'' + ay' + by = 0$,

It is called complementary solution

$$y_c(t) = C_1 y_1 + C_2 y_2.$$

Step ②: to deal with $g(t)$,

Method A: Method of Undetermined Coefficient

This method can only deal with $g(t)$ of the following form:

| $g(t)$ | $Y(x)$ |
|---|---|
| $a_n x^n + a_{n-1}x^{n-1} + \dots + a_0$ | $D_n x^n + D_{n-1}x^{n-1} + \dots + D_0$ |
| $ae^{bx}$ | $De^{bx}$ |
| $a_1 \sin b_1 x + a_2 \cos b_2 x$ | $D_1 \sin b_1 x + D_2 \cos b_2 x$ |

we assume $Y(t)$ to be the form corresponding to different form of $g(t)$, then sub in $L(Y(t)) = g(t)$, comparing both sides, we can find out the constants $D_i$.

Method ③: variation of Parameters

we can use this method to deal with all forms of $g(t)$ (cts to have a unique solution),

Thm Suppose $p, q, g$ are cts in ②, and $y_1, y_2$ be found in step ① (must be linearly independent), then

$$Y(t) = -y_1 \int_{t_0}^{t} \frac{y_2(s)g(s)}{W(s)}\,ds + y_2 \int_{t_0}^{t} \frac{y_1(s)g(s)}{W(s)}\,ds.$$

Step ③ : the solution to $L(y)=g(t)$

is $y = c_1 y + c_2 y + Y(t)$.

Question :

① : Consider $y_1(t) = \begin{cases} t^2 & t \le 0 \\ 0 & t > 0 \end{cases}$, $y_2(t) = \begin{cases} 0 & t \le 0 \\ t^2 & t > 0 \end{cases}$

Show that

a $W(y_1, y_2)(t) = 0$

b $\{y_1, y_2\}$ are linearly independent on $(-\infty, \infty)$

c Can $y_1, y_2$ be 2 solution of a 2nd linear ODE.

Ans: ⓐ

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{cases} \begin{vmatrix} t^2 & 0 \\ 2t & 0 \end{vmatrix} = 0 & t \le 0 \\ \\ \begin{vmatrix} 0 & t^2 \\ 0 & 2t \end{vmatrix} = 0 & t > 0 \end{cases}$$

ⓑ $k_1 y_1 + k_2 y_2 = 0$,

If $t \le 0$, $k_1 t^2 = 0 \Rightarrow k_1 = 0$

Similarly, if $t > 0$, then $k_2 = 0$,

they linearly independent !!

c So Suppose they are the solution of a 2nd linear ODE, then by thm, $y_1, y_2$ are linearly independent iff $W \neq 0$, But by a, b, there is a contradiction!!

furthermore, $y_1, y_2 \in C^1$, they are not 2-differentiable.

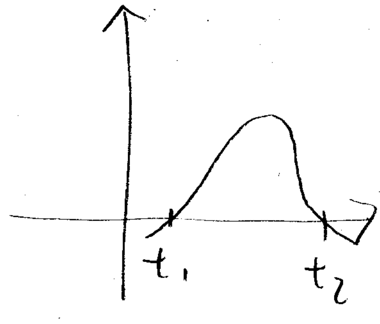
② If $y_1, y_2$ are linearly independent solution of $y'' + p(t)y' + q(t)y = 0$, Show that between consecutive zeros of $y_1$, $\exists!$ zero of $y_2$.

Ans: Assume 2 consecutive zeros of $y_1$ are $t_1, t_2$, i.e. $y_1(t_1) = y_1(t_2) = 0$. $t_1 < t_2$.

then, by cts of $y_1$, $y_1 < 0$ or $y_1 > 0$ in $(t_1, t_2)$.

Suppose $y_1(t) > 0$ $\forall t \in (t_1, t_2)$.

So $y_1'(t_1) \geq 0$ and $y_1'(t_2) \leq 0$



$W(y_1, y_2) = y_2' y_1 - y_1' y_2 \neq 0 \ \forall t \in \mathbb{R}$ as $y_1, y_2$ are linearly independent.

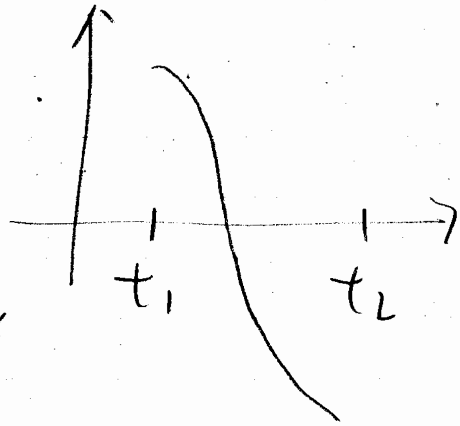
So $W = Ce^{-\int p} > 0$ or $< 0 \ \forall t \in \mathbb{R}$

If $W > 0$,

$W(t_1) = -y_1'(t_1) y_2(t_1) > 0$
$W(t_2) = -y_1'(t_2) y_2(t_2) > 0$

So $y_2(t_1) > 0$ and $y_2(t_2) < 0$,

$\exists$ zero of $y_2$




Other cases are left to you.

If $\exists$ 2 zeros of $y_2$ between $t_1, t_2$.

Say $t_1 < t_3 < t_4 < t_2$, By above argument, interchange $y_1, y_2$, $\exists$ a zero of $y_1$ between $t_1, t_3$

Contradicts with our assumption.

③ Reduction of order:

find the solution of $t^2 y'' - 4ty' + 6y = 0, t > 0$

given that $y_1(t) = t^2$ is a solution.

let $y_2 = t^2 v$, $y_2' = 2tv + t^2 v'$ $y_2'' = 2v + 2tv' + 2tv' + t^2 v''$

$$(2v + 4tv' + t^2 v'')t^2 - 4t(2tv + t^2 v') + 6t^2 v = 0$$

$$t^4 v'' = 0$$

$$v = C_1 t + C_2$$

$$\therefore \quad y_2 = vt^2$$

$$= t^2(C_1 t + C_2) = C_1 t^3 + C_2 t^2.$$

So $y = C_1 t^3 + C_2 t^2$ is the solution.

4 Solve $y'' + 2y' + 5y = 3\sin 2t$

Ans: $r^2 + 2r + 5 = 0$,

$$y_c(t) = e^{-t}(c_1 \cos 2t + c_2 \sin 2t)$$

let $Y(t) = A \sin 2t + B \cos 2t$

$$Y'(t) = 2A \cos 2t - 2B \sin 2t$$

$$Y''(t) = -4A \sin 2t - 4B \cos 2t$$

$$(A + 4B)\cos 2t + (-4A + B)\sin 2t = 3\sin 2t$$

$$A = \frac{3}{17}, \quad B = -\frac{12}{17}$$

So $y = e^{-t}(c_1\cos 2t + c_2\sin 2t) + \frac{3}{17}\sin 2t - \frac{12}{17}\cos 2t$

⑤ Solve $y'' + y = 2\tan t \quad 0 < t < \frac{\pi}{2}$

Ans: $r^2 + 1 = 0$

So $y_c(t) = c_1\cos t + c_2\sin t$.

$$W = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = 1$$

$$Y(t) = -\cos t \int \frac{\sin t(2\tan t)}{1} + \sin t \int \frac{\cos t(2\tan t)}{1}$$

$$= -2\cos t \int \sec t - \cos t - 2\sin t\cos t$$

$$= -2\cos t \int \sec t$$

$$= -2\cos t \ln(\sec t + \tan t)$$

$$\therefore \ y = c_1\cos t + c_2\sin t - 2\cos t\ln(\sec t + \tan t).$$